Chapter 5: Integrals

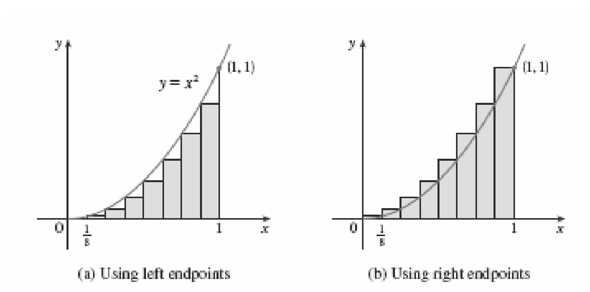
Section 5.1: Areas and Distances

The Area Problem

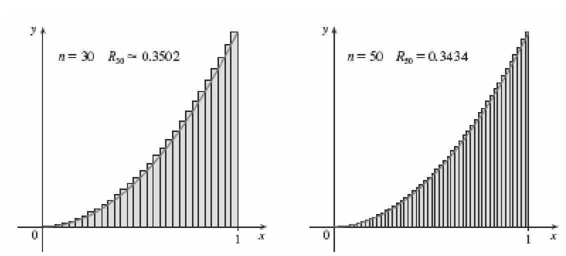
When finding the area of an object that consists of straight edges, it's easy to find its area. A rectangle's area is simply the product of its length and width, but when trying to find the area under a polynomial function, the problem becomes much more difficult.

Riemann Sums

We can agree that a close approximation of the area under a curve could be given by using rectangles that change in height with the function as shown below.



As the amount of rectangles used to approximate is increased, the area becomes closer and closer to the exact value of the area.



Therefore, as the amount of rectangles (n) approaches infinity, we get the exact value of the area.

Definition:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}[f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x]$$

Section 5.2: The Definite Integral

In a simpler fashion, the formula above is made into the Definite Integral equation.

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

Since Riemann Sums of left hand and right hand approximations lead the ending value to be greater or less than the real area, we use the midpoint rule in which the midpoints of the rectangle at each point is sampled for an area.

<u>Properties of the Definite Integral</u>

Properties of the Integral

1. $\int_a^b c\,dx = c(b-a)$, where c is any constant
2. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant
4. $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

Section 5.3: The Fundamental Theorem of Calculus

The fundamental theorem of Calculus essentially branches the two main forms of calculus, differential and integral calculus.

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

It essentially describes integrating and deriving a function as inverse functions of each other.

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

This definition allows one to solve an integral by hand rather than by a calculator.

Section 5.4: Indefinite Integrals

An indefinite integral is any integral in which the limits are not defined in this case, the exact position of the graph of the function is also not defined. To cope with this a “dummy” variable of C is placed after the integrated function.

1 Table of Indefinite Integrals

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

When an integral does have limits, the net change theorem is used to solve it.

Net Change Theorem:

$$\int_a^b F'(x)dx = F(b) - F(a)$$

Section 5.5: The Substitution Rule

If u = g(x) is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Also, its helpful to know the simple symmetric function equations to do identifiable problems quickly.

7 Integrals of Symmetric Functions Suppose f is continuous on $[-a, a]$.

(a) If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 0$.

Section 5.6: Logarithm Defined as an Integral

The Natural Log Function is defined by:

$$\ln x = \int_1^x \frac{1}{t}dt$$

To deal with integrals and their logarithms, there are a couple of review functions that we need:

3 Laws of Logarithms If x and y are positive numbers and r is a rational number, then

1. $\ln(xy) = \ln x + \ln y$ **2.** $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$ **3.** $\ln(x^r) = r\ln x$

11 Laws of Exponents If x and y are real numbers and r is rational, then

1. $e^{x+y} = e^x e^y$ **2.** $e^{x-y} = \frac{e^x}{e^y}$ **3.** $(e^x)^r = e^{rx}$

Finally, the way to integrate power functions:

$$\frac{d}{dx}(a^x) = a^x \ln a$$