Chapter 4 Notes

Section 4.1 Maximum and Minimum Values

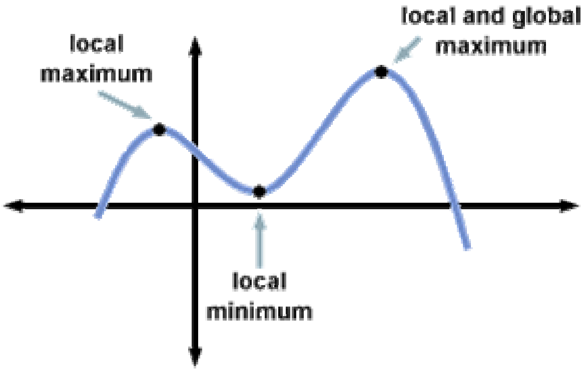
- A function *f* has an **absolute (global) maximum** at *c* if $f(c) \geq f(x)$ for all *x* in the domain of *f*. The number $f(c)$ is called the maximum value of *f* on the domain.

- A function *f* has an **absolute (global) minimum** at *c* if $f(c) \leq f(x)$ for all *x* in the domain of *f*. The number $f(c)$ is called the minimum value of *f* on the domain.

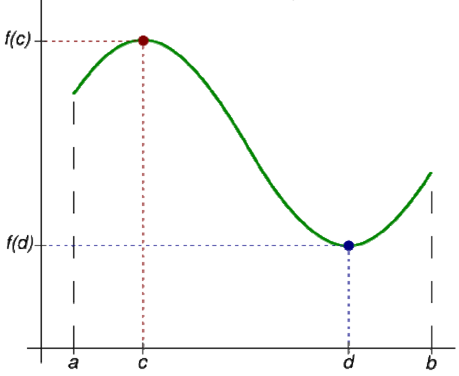
NOTE: The max. and min. values of *f* are called the extreme values of *f*. Also, the value of the max. or min. is simply the y-coordinate.

- A function has a **local (relative) maximum** at *c* if $f(c) \leq f(x)$ when *x* is near *c*. *f* has a **local (relative) minimum** at *c* if $f(c) \leq f(x)$ when *x* is near *c*.

Graphically:



- **<u>Extreme Value Theorem</u>**: If *f* is continuous on a closed interval [a,b] then *f* has an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some number *c* and *d* on [a,b].



- **<u>Fermat's Theorem</u>**: If *f* has a local max. or min. at *c* and if $f'(c)$ exists, then $f'(c) = 0$.
- **<u>Definition of a Critical Number</u>**: A critical number of a function *f* is a number *c* in the domain of *f* such that $f'(c) = 0$ or f DNE.

The Closed Interval Method

To find the absolute max. or min. values of a continuous function *f* on a closed interval [a,b]…

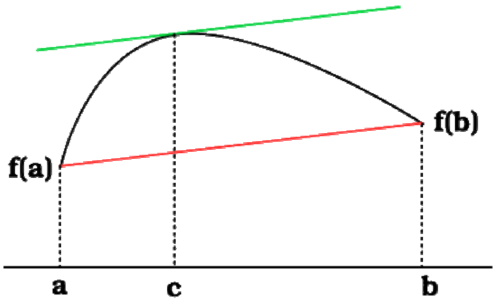
1. Find the values of *f* at the critical numbers of *f* in (a,b)
2. Find the values of *f* at the endpoints of the interval
3. The largest of all these values is the absolute maximum value
   The smallest of all these values is the absolute minimum value

Section 4.2 The Mean Value Theorem

- **Rolle's Theorem**

  Let *f* be a function that satisfies the following:
    1) *f* is continuous on [a,b]
    2) *f* is differentiable on (a,b)
    3) $f(a)=f(b)$

  Then there exists a number c in (a,b) such that $f'(c)=0$.



- **The Mean Value Theorem**

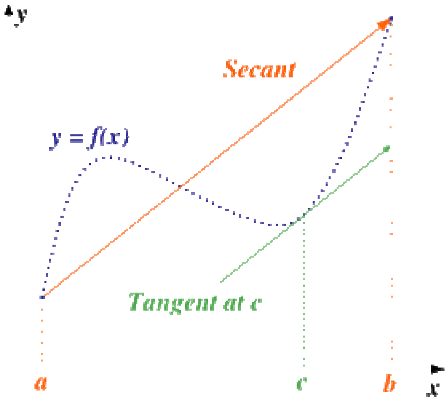
  Let *f* be a function that satisfies the following hypothesis:
    1) *f* is continuous on [a,b]
    2) *f* is differentiable on (a,b)

  Then there exists a number c such that in (a,b) such that

$$\boxed{f'(c)=\frac{f(b)-f(a)}{b-a}}$$

Think: $f(b)-f(a)=f'(c)(b-a)$ point-slope form

Graphically:



Section 4.3 How Derivatives Affect the Shape of a Graph

- Increasing/Decreasing Test
  - a.) If $f'(x) > 0$ on an interval, then *f* is increasing on that interval
  - b.) If $f'(x) < 0$ on an interval, then *f* is decreasing on that interval

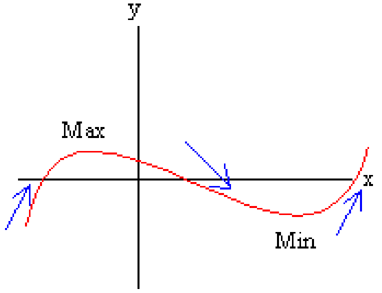
- **The First Derivative Test**

  Suppose that *c* is a critical number of a continuous function *f*.

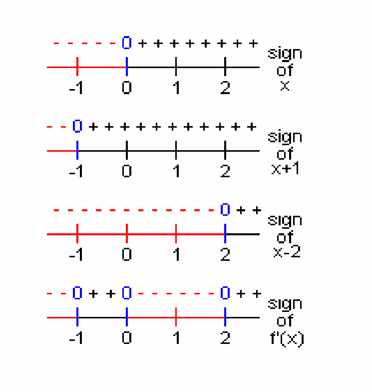
  1.) If f' changes from positive to negative @ *c* then *f* has a local maximum @ *c*.

  2.) If f' changes from negative to positive @ *c* then *f* has a local minimum @ *c*.

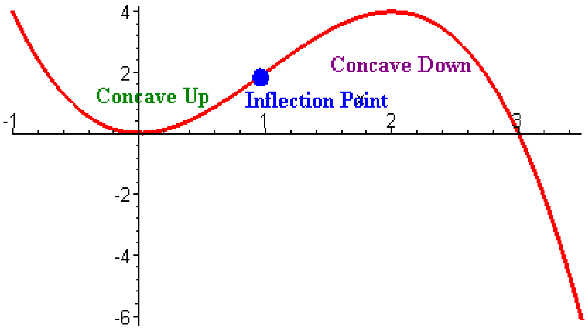
  3.) If f' does not change sign @ *c*, then *f* has no local maximum or minimum @ *c*. (relates to saddle points)



- The first derivative test applies the change in sign of the slope of a graph to locate maximum and minimum values of the function. We use sign charts to test this change in sign on the intervals that the function's critical numbers form.



- Concavity Based on the 2nd Derivative
  - The points where concavity changes are called inflection points or points of inflection (POI)
  - If $f''(x) > 0$ on an interval, then *f* is concave up
  - If $f''(x) < 0$ on an interval, then *f* is concave down



- **The 2^{nd} Derivative Test**

  Suppose f'' is continuous near *c*

  a) If $f'(c)=0$ and $f''(c)>0$, then *f* has a local min. @ *c*.

  b) If $f'(c)=0$ and $f''(c)<0$, then *f* has a local max. @ *c*.

  - If horizontal tangent and concave up=local minimum
  - If horizontal tangent and concave down=local max

  We can use the 2^{nd} derivative test as an alternative way to locate local max.'s and min.'s. The premise of this test also allows us to test concavity of a function by applying sign chart tests to the intervals formed by POI's.

Section 4.4 L'Hospital's Rule

Suppose *f* and *g* are differentiable and $g'(x)\neq 0$ near *a* (except possibly at *a*).

Suppose that: $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$

Or that: $\lim_{x\to a} f(x)=\pm\infty$ and $\lim_{x\to a} g(x)=\pm\infty$

Then: $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$

L'Hospital's rule gives us an alternative way of solving normally unsolvable limits by manipulating functions as long as they are in indeterminate forms. The indeterminate forms are as follows:

$\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$, $0\bullet\infty$, 0^0 , ∞^0 , 1^∞

Section 4.5 Curve Sketching

Curve sketching involves 8 sequential steps in order to accurately plot the graph of a function. The steps are as follows:

1) **Domain**- find the domain of the function
2) **Intercepts**- find the x & y intercepts of the function by setting x=0 and y=0
3) **Symmetry**- to find symmetry:*If $f(-x)=f(x)$, then the function is even and symmetric about the y-axis
   *If $f(-x)=-f(x)$, then the function is odd and symmetric about the origin
   *If neither is true, the function has no symmetry
4) **Asymptotes**- find the vertical and horizontal asymptotes by testing the limits of the intervals of the domain
5) **Intervals of Increasing and Decreasing**- apply the first derivative test to find the intervals where the function is increasing and decreasing on
6) **Local Minimum and Maximum Values**- from the sign chart in step 5, determine local max's and min's and plug them into the function to find their values
7) **Cocavity and Points of Inflection**- apply the second derivative test to find the intervals of concavity and determine POI's from the test's sign chart
8) **Sketch the Curve**

- The 8 steps can be made into a helpful acronym **DISAILCS** (DIE-SAIL-SEE-ES)

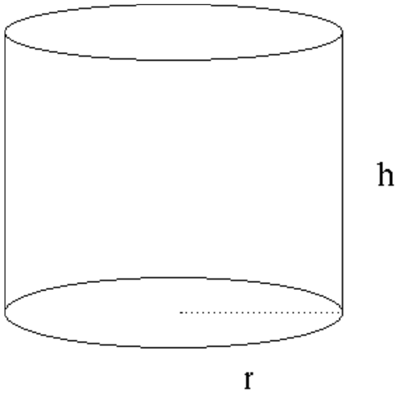
Section 4.7 Optimization Problems

In optimization problems, we are looking to absolutely maximize or minimize the values which a function can possess. The biggest challenge in optimization problems is to

Steps for Solving Optimization Problems
1. Understand the Problem The first step is to read the problem carefully until it is clearly
understood. Ask yourself: What is the unknown? What are the given quantities? What are the given conditions?
2. Draw a Diagram In most problems it is useful to draw a diagram and identify the given and required quantities on the diagram.
3. Introduce Notation Assign a symbol to the quantity that is to be maximized or minimized
(let's call it **Q** for now). Also select symbols (a,b,c,…,x,y) for other
unknown quantities and label the diagram with these symbols. It may help to use initials as suggestive symbols—for example, **A** for area, **h** for height, **t** for time.
4. Express ***Q*** in terms of some of the other symbols from Step 3.
5. If ***Q*** has been expressed as a function of more than one variable in Step 4, use the given information to find relationships (in the form of equations) among these variables. Then use these equations to eliminate all but one of the variables in the expression for ***Q***. Thus, ***Q*** will be expressed as a function of *one* variable ***x***, say, ***Q*= *f*(x)**. Write the domain of this function.
6. Use the methods of Sections 4.1 and 4.3 to find the *absolute* maximum or minimum value of ***f***. In particular, if the domain of ***f*** is a closed interval, then the
Closed Interval Method in Section 4.1 can be used.

First Derivative Test for Absolute Extreme Values
Suppose that ***c*** is a critical number of a continuous function ***f*** defined on an interval.
(a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of ***f***.
(b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of ***f***.

Let variable *r* be the radius of the circular base and variable *h* the height of the cylinder.



The total surface area of the cylinder is given to be

$$3\pi = \text{(area of base)} + \text{(area of the curved side)} = \pi r^2 + (2\pi\, r)h,$$

so that $2\pi rh = 3\pi - \pi r^2$ or $h = \dfrac{3\pi - \pi r^2}{2\pi r} = \dfrac{3}{2r} - \dfrac{1}{2}r$

We wish to MAXIMIZE the total VOLUME of the cylinder

$$V = \text{(area of base) (height)} = \pi r^2 h \,.$$

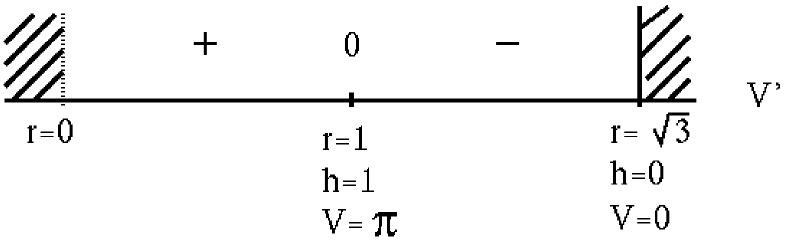
However, before we differentiate the right-hand side, we will write it as a function of r only. Substitute for h getting

$$V = \pi r^2 h = \pi r^2 (\frac{3}{2r} - \frac{1}{2} r) = \frac{3}{2}\pi r - \frac{1}{2}\pi r^3$$

Now differentiate this equation, getting: $V' = \frac{3}{2}\pi - \frac{1}{2}\pi 3r^2 = \frac{3}{2}\pi(1-r^2)$

$= \frac{3}{2}\pi(1-r)(1+r) = 0$ for $r = 1$ or $r = -1$.

But $r \neq -1$ since variable r measures a distance and $r > 0$. Since the base of the box is a circle and there is 3π ft.^{2} of material, it follows that $0 < r \leq \sqrt{3}$. See the adjoining sign chart for V' .



If r =1 ft. and h=1 ft.,

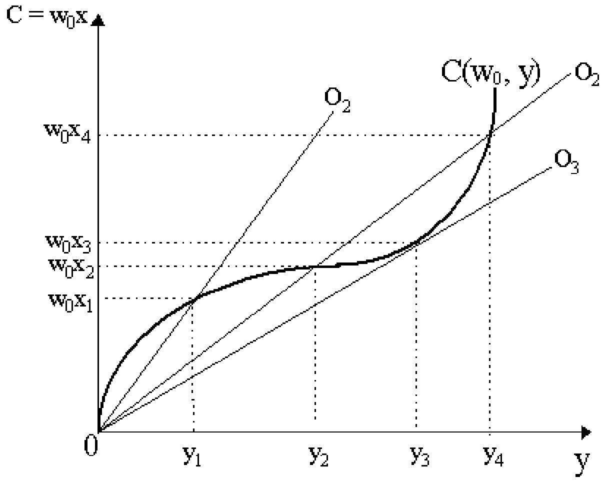
then r =1 ft. and h=1 ft.

$V = \pi$ ft.^{3}

is the largest possible volume of the cylinder.

Section 4.8 Applications to Business and Economics

From section 3.3, the **cost function** $C(x)$ was defined as the cost of producing x units of a certain production, then the **marginal cost** was defined as the rate of change, or derivative, of C with respect to x. Therefore, the marginal cost can be represented as $C'(x)$. Graphically, the cost function looks something like this:

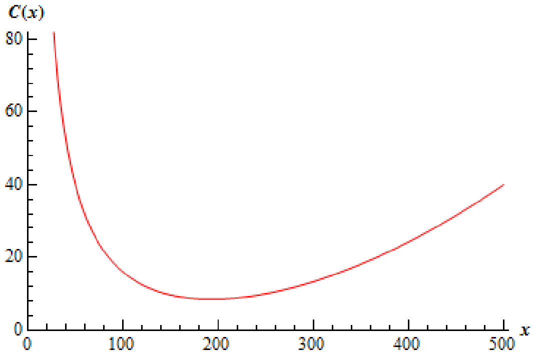


In Section 4.8, we are given the **average cost function** defined as

$$c(x) = \frac{C(x)}{x}$$

and represents the cost per unit when x units are produced.

The average cost function graphically:



If we notice that the graph appears to have an absolute minimum, we can local the critical point of c by using the Quotient Rule to differentiate the average cost function. Therefore:

$$c'(x) = \frac{xC'(x) - C(x)}{x^2}$$

Since $c'(x) = 0$ when $xC'(x) - C(x) = 0$, then $C'(x) = \frac{C(x)}{x} = c(x)$

Therefore: **if the average cost is a minimum, then marginal cost = average cost**. Thus, we can use principles of optimization to minimize average cost.

Considering marketing, $p(x)$ can be the price per unit that the company can charge if it sells x units. Then p is the **demand function** (or **price function**) expected to be a decreasing function of **x**.

If x units are sold, then the total revenue is

$$R(x) = xp(x)$$

and R is the **revenue function** (or **sales function**). R' of the revenue function is called the **marginal revenue function** and is the rate of change, or derivative, of revenue with respect to number of units sold.

With x units sold, the total profit is

$$P(x) = R(x) - C(x)$$

and P is the **profit function.** The **marginal profit function** is P'. To maximize profit, we look for the critical numbers of P where the marginal profit is 0, and if

$P'(x) = R'(x) - C'(x) = 0$

then $R'(x) = C'(x)$

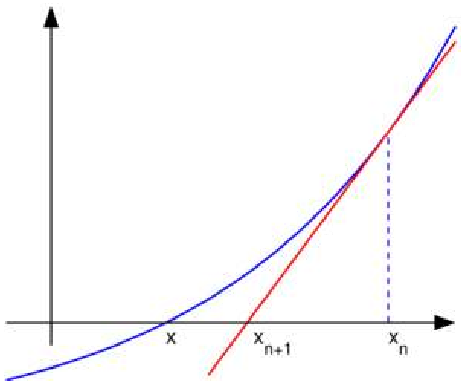
Therefore: **If the profit is a maximum, then marginal revenue = marginal cost**.

Once again, we can use optimization to maximize marginal cost.

Section 4.9 Newton's Method

The idea behind Newton's Method is that since a tangent line is close to the actual shape of a function's curve, then the tangent's x-intercepts also must be close to the roots of the curve that we are trying to find. Since the tangent line is a line, it is far easier to find its x-intercept, and if we can find a tangent line extremely close to the root of the actual curve, then we can derive a reasonable estimation of the curve's actual root.

Graphically:



To find a formula for the tangent line, in this case x_{n+1}, in terms of x_n, we use the fact that the slope of the tangent is $f'(x_{n+1})$, so its equation is

$$y - f(x_n) = f'(x_n)(x - x_n)$$

Since the x-intercept of the tangent is x_{n+1}, setting y=0 obtains $0 - f(x_n) = f'(x_n)(x_2 - x_1)$

and solving for x_{n+1} yields $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

To yield the most exact results, this process must be carried out to an infinite extent so it will converge at the wanted root.

Therefore: $$\lim_{n\to\infty} x_n = r$$

Section 4.10 Antiderivatives

<u>Definition of an antiderivative</u>: A fuction *F* is called an **antiderivative** of *f* on an interval *I* if $F'(x) = f(x)$ for all *x* in *I*

In its most general form, an antiderivative can be defined by the following theorem:

- If *F* is an antiderivative of *f* on an interval *I*, then the most general antiderivative of *f* on *I* is

$$\boxed{F(x)+C}$$

where C is an arbitrary constant

<u>Common antiderivatives</u>:

Function	**Antiderivate**
$f(x) = A$	$F(x) = Ax + C$
$f(x) = x$	$F(x) = \frac{1}{2}x^2 + C$
$f(x) = x^n$	$F(x) = \frac{1}{n+1}x^{n+1} + C$
$f(x) = 7x^3 + 3x - 4$	$F(x) = \frac{7}{4}x^4 + \frac{3}{2}x^2 - 4x + C$
$f(x) = \frac{1}{x}$	$F(x) = \ln\lvert x\rvert + C$
$f(x) = \sin x$	$F(x) = -\cos x + C$
$f(x) = \cos x$	$F(x) = \sin x + C$
$f(x) = \sec x \tan x$	$F(x) = \sec x + C$
$f(x) = \frac{-1}{\sqrt{1-x^2}}$	$F(x) = \cos^{-1}(x) + C$